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Stabilization of Concentration Profiles in Catalyst Particles

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INTRODUCTION

In a recent paper [1], Aronson and Peletier studied the global stability of concentration profiles in a one-dimensional model of a catalyst particle. They considered an infinite slab of homogeneous material with catalytic material situated on both of its faces. The slab is immersed in a bath in which the concentration of the reactant is kept at a constant value. This led to the study of the following mixed initial-boundary value problem

$$u_t = u_{xx}, \qquad \text{for} \quad 0 < x < 1, \quad t > 0, \tag{1}$$

$$u_x(i, t) = (-1)^i \lambda f(u(i, t)), \qquad \text{for} \quad i = 0, 1, \quad t > 0, \tag{2}$$

$$u(x, 0) = \psi(x), \qquad \text{for} \quad 0 \leqslant x \leqslant 1. \tag{3}$$

Here x denotes the spatial coordinate perpendicular to the faces of the slab, these being situated at $x = 0$ and $x = 1$. The variable t denotes time, u denotes a dimensionless concentration and λ denotes a positive parameter. The function f appearing in the boundary conditions is given by

$$f(u) = [k_1 u/(1 + k_2 u)^2] + u - 1,$$

in which k_1 and k_2 are suitably chosen positive constants. This function is related to the rate of consumption of the reactant by the catalytic material situated on the faces.

It was shown in [1] that for each $\psi \in C([0, 1])$ problem (1)–(3) has a unique solution $u(x, t; \psi)$. It was also shown that, depending on the value of λ, problem (1), (2) could have 3, 5, 7, or 9 equilibrium solutions. The main emphasis in [1] was on a discussion of the question of stability of these equilibrium solutions. Specifically, if $\bar{u}(x)$ is such an equilibrium solution, a partial characterization was given of the region of attraction $A(\bar{u})$, where

$$A(\bar{u}) \overset{\text{def}}{=} \{\psi \in C([0, 1]): u(x, t, \psi) \to \bar{u}(x) \text{ uniformly on } [0, 1] \text{ as } t \to \infty\}.$$

In this paper we shall be interested in the following question. Given any $\psi \in C([0, 1])$, must $u(x, t; \psi)$ converge as $t \to \infty$ to some equilibrium solution? That is, is it true that

$$\bigcup_{j=1}^{n} A(\bar{u}_j) = C([0, 1]),$$

where $\{\bar{u}_j\}$, $j = 1, 2, \ldots, n$, is the set of equilibrium solutions? We shall show that this is indeed the case, and that in addition convergence to the relevant equilibrium solution holds in $C^1([0, 1])$.

In [1] the value of λ and the form of the function f played an important role in the characterization of the equilibrium solutions and their regions of attraction. In the present paper we shall not be interested in a detailed description of the equilibrium solutions. It will therefore be possible to obtain without extra effort the above result for the more general problem

$$u_t = u_{xx}, \qquad 0 < x < 1, \quad t > 0, \tag{4}$$

$$u_x(i, t) = (-1)^i f_i(u(i, t)), \qquad i = 0, 1, \quad t > 0, \tag{5}$$

$$u(x, 0) = \psi(x), \qquad 0 \leqslant x \leqslant 1, \tag{6}$$

in which f_0 and f_1 are twice continuously differentiable functions defined on $\mathbb{R}$, each satisfying the following hypotheses:

(H1) There exists a positive constant $a < \infty$ such that

$$sf(s) > 0 \qquad \text{for} \quad |s| > a.$$

As we shall see, this condition ensures that

(i) problem (4), (5) has at least one equilibrium solution;

(ii) $u(x, t; \psi)$ is uniformly bounded for $0 \leqslant x \leqslant 1$, $t \geqslant 0$.

(H2) The equilibrium solutions of problem (4), (5) are isolated in $C([0, 1])$.

To prove our result we will use the invariance principle discovered for ordinary differential equations by LaSalle and extended to general semigroups by Hale [9]. The earliest example of the use of similar techniques to study the asymptotic behavior of a partial differential equation seems to be the work of Zelenyak [11]. Invariance techniques have now been successfully applied to a number of problems involving partial differential equations (see, for example, [2, 8]). In particular, a result similar to ours for solutions of a one-dimensional semilinear parabolic equation with zero Dirichlet data at the lateral boundary has been established by Rudenko [10], who used a result due to Zelenyak [12], and by Chafee and Infante [5]. Recently the same problem with zero Neumann data was treated by Chafee [4] who has also studied in [3] a related problem on an infinite interval.

The major burden of our work is to show that the solution $u(x, t; \psi)$ has sufficient regularity properties for the invariance principle to be applied. In particular, we show that if $\psi \in C([0, 1])$ then $u_t(., .; \psi) \in C([0, 1] \times [a, b])$ for $0 < a < b < \infty$. This is done in Section I by considering the equivalent system of Volterra integral equations and by use of the maximum principle for the heat equation. Then in Section II we fairly rapidly prove the main result.

I

We first introduce some notation. We shall write

$$Q_T = \{(x, t) : 0 < x < 1, 0 < t \leqslant T\},$$
$$S_T = \{(x, t) : x \in \{0, 1\}, 0 < t \leqslant T\},$$

where $T > 0$ may be infinite. Let $Q = Q_\infty$, $S = S_\infty$. Denote by $\bar{Q}_T$ the closure of Q_T.

A function $u = u(x, t; \psi)$ is said to be a solution of problem (4)–(6) if $u \in C(\bar{Q})$, $u_x \in C(Q \cup S)$, $u_{xx} \in C(Q)$, $u_t \in C(Q)$, and (4)–(6) hold. For problem (1)–(3), which can be reduced to a special case of (4)–(6), existence and uniqueness was established in [1]. However, examination of the proof reveals that the only properties of the function f in (2) which were needed were that $f \in C^2(\mathbb{R})$ and a property which ensured that $u(x, t; \psi)$ is uniformly bounded in $\bar{Q}$. In problem (4)–(6) it is hypothesis (H1) which takes care of the boundedness of u. To prove this we use a slight variation of a maximum principle established in [1].

LEMMA 1. *Let $v_j(x) = p_j + (q_j - p_j)x$ for $j = 1, 2$. Assume that*

$$q_1 \geqslant p_1 + f_0(p_1), \qquad p_1 \geqslant q_1 + f_1(q_1),$$

and

$$q_2 \leqslant p_2 + f_0(p_2), \qquad p_2 \leqslant q_2 + f_1(q_2).$$

If ψ satisfies

$$v_1 \leqslant \psi \leqslant v_2 \qquad on \quad [0, 1],$$

then

$$v_1(x) \leqslant u(x, t; \psi) \leqslant v_2(x) \qquad in \quad \bar{Q}.$$

For the proof we refer to [1].

Given any $\psi \in C([0, 1])$ there exists a constant u^* satisfying

(i) $u^* \geqslant \max\{a_0, a_1\}$, where a_i is the constant defined in hypothesis (H1) for the function f_i $(i = 0, 1)$, and

(ii) $u^* \geqslant |\psi(x)|$ for all $x \in [0, 1]$.

By Lemma 1 and hypothesis (H1),

$$-u^* \leqslant u(x, t; \psi) \leqslant u^* \qquad \text{in } \bar{Q} \text{ for all } t \geqslant 0. \tag{7}$$

By (7) the asymptotic behavior of $u(x, t; \psi)$ is unaffected by the values of f_i outside $[-u^*, u^*]$.

Thus, since $f_i \in C^2(\mathbb{R})$, without loss of generality we may and shall assume that there are constants M_j $(j = 0, 1, 2)$ such that for $i = 0$ and 1,

$$|f_i(s)| \leqslant M_0, \qquad |f_i'(s)| \leqslant M_1, \qquad |f_i''(s)| \leqslant M_2 \qquad \text{for all} \quad s \in \mathbb{R},$$

where primes denote differentiation.

One can now proceed as in [1] to prove the following result.

THEOREM 1. *Let $\psi \in C([0, 1])$. Then problem (4)–(6) possesses a unique solution. Moreover, for each $T > 0$ there exists a constants C_T such that*

$$\max_{\bar{Q}_T} |u(x, t; \psi_1) - u(x, t; \psi_2)| \leqslant C_T \max_{[0,1]} |\psi_1(x) - \psi_2(x)|$$

for any $\psi_1, \psi_2 \in C([0, 1])$.

Let $u(x, t; \psi)$ be the solution of problem (4)–(6). By means of the following lemma we can reduce our problem to one only involving the two functions $u(0, t)$ and $u(1, t)$. (For convenience we shall sometimes omit reference to ψ.)

LEMMA 2. *Let $u(0, t)$ and $u(1, t)$ belong to $C^1(0, \infty)$. Then $u_t \in C([0, 1] \times [a, b])$ for $0 < a < b < \infty$.*

Proof. Let $G(x, \xi, t)$ be the Green function for the heat equation on $(0, 1) \times (0, \infty)$ with zero Neumann data. It can be given explicitly by

$$G(x, \xi, t) = 1 + 2 \sum_{n=1}^{\infty} e^{-n^2\pi^2 t} \cos n\pi x \cos n\pi\xi.$$

Then the solution u of problem (4)–(6) can be written as

$$u(x, t) = \int_0^t G(x, \xi, t)\, \psi(\xi)\, d\xi - \sum_{i=0}^{1} \int_0^t G(x, i, t - \tau)\, f_i(u(i, \tau))\, d\tau. \tag{8}$$

To begin with, we assume that $u(i, t) \in C^1([0, \infty))$ for $i = 0, 1$. We substitute $\tau = t - s$ in (8) and differentiate with respect to t. This yields

$$u_t(x, t) = \int_0^1 G_t(x, \xi, t)\, \psi(\xi)\, d\xi - \sum_{i=0}^{1} G(x, i, t) f_i(u(i, 0))$$
$$- \sum_{i=0}^{1} \int_0^t G(x, i, s) f_i'(u(i, t-s))\, u'(i, t-s)\, ds.$$

If $0 < a < b < \infty$ we can use the expression for G to write the first term as a uniformly convergent series of functions in $C([0, 1] \times [a, b])$. Hence this term belongs to $C([0, 1] \times [a, b])$.

The second term clearly belongs to $C([0, 1] \times [a, b])$. To treat the third term, note that both $f_i'(u(i, t - s))$ and $u'(i, t - s)$ are bounded for $0 \leqslant s \leqslant t$. Moreover, for $x, \xi \in [0, 1]$,

$$\int_0^t | G(x, \xi, t)|\, ds \leqslant t + 2 \sum_{n=1}^{\infty} [(1 - e^{-n^2\pi^2 t})/n^2\pi^2].$$

Hence this term can also be expressed as a uniformly convergent series of functions belonging to $C([0, 1] \times [a, b])$ and therefore belongs itself to $C([0, 1] \times [a, b])$.

Thus we have shown that $u_t \in C([0, 1] \times [a, b])$ if $u'(i, t)$ is continuous up to $t = 0$ for $i = 0$ and 1. It remains to dispose of this last condition. Instead of problem (4)–(6) we consider the problem (4), (5) with initial value

$$\bar{\psi}(x) = u(x, \tau; \psi)$$

for some $\tau > 0$. Since $u(x, t; \bar{\psi}) = u(x, t + \tau; \psi)$ it follows that $u(i, t; \bar{\psi}) \in C^1([0, \infty))$. By the first part of the proof $u_t(., .; \bar{\psi}) \in C([0, 1] \times [a, b])$ for $0 < a < b < \infty$ and hence $u_t(., .; \psi) \in C([0, 1] \times [a, b])$ for $\tau < a < b < \infty$. Since we may choose τ as small as we wish, it follows that $u_t(., .; \psi) \in C([0, 1] \times [a, b])$ whenever $0 < a < b < \infty$.

It follows from (8) that the functions $u_i(t) = u(i, t; \psi)$ satisfy the pair of Volterra integral equations

$$u_i(t) = \int_0^1 G(i, \xi, t)\, \psi(\xi)\, d\xi - \sum_{j=0}^{1} \int_0^t g_{ij}(t - \tau) f_j(u_j(\tau))\, d\tau, \qquad i = 0, 1,$$

where $g_{ij}(t) = G(i, j, t)$. To write these equations more compactly, we introduce the vector-valued functions $u(t) = (u_0(t), u_1(t))$, $f(u) = (f_0(u_0), f_1(u_1))$ and $\phi(t) = (\phi_0(t), \phi_1(t))$, where $\phi_i(t) = \int_0^1 G(i, \xi, t)\, \psi(\xi)\, d\xi$, and the matrix $G(t) = (g_{ij}(t))$. We then obtain

$$u(t) = \phi(t) - \int_0^t G(t - \tau) f(u(\tau))\, d\tau. \tag{9}$$

Let I be an interval on the real line, and let $\mathscr{C}^k(I)$ be the set of functions $I \to \mathbb{R}^2$ which, together with their first k derivatives, are continuous on I. In [1] it was shown that (9) has a unique solution u, which for every $T > 0$ belongs to $\mathscr{C}([0, T])$. It follows from Lemma 2 that it will be enough to prove that in addition u belongs to $\mathscr{C}^1(0, T)$. It will be helpful to prove the following lemma, which is also of critical importance for the analysis in Section II.

LEMMA 3. *Let $\psi \in C([0, 1])$ and let $\delta > 0$. There exists a constant $k > 0$ such that $|u_x(x, t; \psi)| \leqslant k$ for all $x \in [0, 1]$ and $t \geqslant \delta$.*

Proof. We have already shown that $u(x, t; \psi)$ is uniformly bounded on $\bar{Q}$, and thus on S. In view of the boundary conditions (5) this implies that u_x is bounded on S. Moreover, since $\delta > 0$, $u(., \delta; \psi) \in C^1([0, 1])$. Thus u_x is bounded on the parabolic boundary of the cylinder $[0, 1] \times [\delta, \infty)$. Since u_x satisfies the heat equation in Q, it follows by the maximum principle that u_x is bounded in $[0, 1] \times [\delta, \infty)$ as required.

Next we estimate the behavior of $\phi(t)$ and $\phi'(t)$. Because $G(x, \xi, t)$ is singular at $t = 0$, we may also expect $\phi'(t)$ to be singular at $t = 0$. Let

$$J(x, t) = [1/2(\pi t)^{1/2}]\, e^{-x^2/4t}, \qquad x \in \mathbb{R},\ t > 0,$$

denote the source function for the heat equation in one dimension.

Let $\psi \in C([0, 1])$ and let $\tilde{\psi}$ denote the unique extension of ψ to $\mathbb{R}$ which is symmetric with respect to $x = 0$ and $x = 1$. Then

$$\int_0^1 G(x, \xi, t)\, \psi(\xi)\, d\xi = \int_{-\infty}^{\infty} J(x - \xi, t)\, \tilde{\psi}(\xi)\, d\xi, \qquad x \in [0, 1],\ t > 0. \quad (10)$$

It follows that $\phi \in C([0, \infty))$, and that

$$|\phi_i(t)| \leqslant \max_{[0,1]} |\psi(x)|, \quad t \geqslant 0, \qquad i = 0, 1. \quad (11)$$

Also, if $\psi \in C^1([0, 1])$,

$$\phi_i'(t) = -\int_{-\infty}^{\infty} J_\xi(i - \xi, t)\, \tilde{\psi}'(\xi)\, d\xi,$$

and thus, by a routine computation,

$$|\phi_i'(t)| \leqslant [1/(\pi t)^{1/2}] \max_{[0,1]} |\psi'(x)|, \qquad t > 0, \qquad i = 0, 1. \quad (12)$$

We also need an estimate for the behavior of the functions $g_{ij}(t)$ as $t \to 0+$. From (10) we see that

$$G(x, \xi, t) = \sum_{n=-\infty}^{\infty} \{J(x - \xi - 2n, t) + J(x + \xi - 2n, t)\}.$$

Hence

$$|g_{ij}(t)| \leqslant \omega(t), \qquad t > 0,$$

where

$$\omega(t) \equiv G(0, 0, t) = 2 \sum_{n=-\infty}^{\infty} J(-2n, t).$$

An elementary computation now shows that (i) $\omega(t)$ is continuous and non-increasing for $t > 0$, and (ii) $\omega(t) \sim (\pi t)^{-1/2}$ as $t \to 0+$.

For future reference we introduce three functions $h_i(t)$ $i = 1, 2, 3$ which are related to $\omega(t)$. Let $\alpha \in (\frac{1}{2}, 1)$ and

$$h_1(t) = \int_0^t \omega(s)\, ds, \qquad \text{for} \quad t > 0,$$

$$h_2(t) = \sup_{(0,t)} s^\alpha \omega(s), \qquad \text{for} \quad t > 0,$$

$$h_3(t) = \sup_{(0,t)} s^\alpha \int_0^s \omega(s - r)\, r^{-\alpha}\, dr, \qquad \text{for} \quad t > 0.$$

It is clear from properties (i) and (ii) of ω that the functions h_i are well defined. Moreover, $h_i \to 0$ for $t \to 0+$ and $i = 1, 2, 3$. The value of α will be fixed throughout our discussion.

In view of the singular behavior of ϕ' we shall discuss (9) in a weighted space of continuous functions.

Definition. Let $\gamma > 0$. We denote by $X(\gamma)$ the space of functions $\zeta \in \mathscr{C}([0, \gamma]) \cap \mathscr{C}^1((0, \gamma])$ such that

$$\| \zeta \|_X = \sum_{i=0}^{1} \{\sup_{(0,\gamma)} | \zeta_i(t)| + \sup_{(0,\gamma)} | t^\alpha \zeta_i'(t)|\} < \infty.$$

It is not difficult to show that $(X(\gamma), \| \cdot \|_X)$ is a Banach space. We shall frequently omit reference to γ.

Theorem 2. *Let* $\psi \in C([0, 1])$. *Then* $u(t) = (u_0(t), u_1(t))$ *belongs to* $\mathscr{C}^1(0, T)$ *for every* $T > 0$.

Proof. By an argument similar to that at the end of the proof of Lemma 2 it is clear that without loss of generality we may suppose that $\psi \in C^1([0, 1])$. Now define the operator A by

$$(Au)(t) \equiv \int_0^t G(t - \tau) f(u(\tau))\, d\tau.$$

Then (9) can be written as

$$u = \phi - Au.$$

We shall first show that the operator

$$Ku = \phi - Au$$

is a contraction on a certain closed ball of $X(\gamma)$ for γ sufficiently small. The existence and uniqueness of a solution of (9) in $X(\gamma)$ then follows from the well-known Banach fixed point theorem. We will then show how this argument may be repeated to show that $u(t) \in \mathscr{C}^1(0, T)$ for any $T > 0$.

Since $\alpha > \frac{1}{2}$, it follows from (12) that $\phi \in X$. We now show that because $\alpha < 1$ the operator A is defined and bounded on X. Let $u \in X$. Then we have for $i = 0, 1$ and $t \in (0, \gamma)$

$$|(Au)_i(t)| = \left| \sum_{j=0}^{1} \int_0^t g_{ij}(t - \tau) f_j(u_j(\tau))\, d\tau \right| \leqslant 2M_0 h_1(\gamma),$$

and hence

$$\| Au \|_{\mathscr{C}} \leqslant 4M_0 h_1(\gamma),$$

where

$$\| \zeta \|_{\mathscr{C}} = \sum_{i=1}^{1} \sup_{(0,\gamma)} | \zeta_i(t)|.$$

Moreover, it follows after a straightforward computation that for $i = 0, 1$:

$$(Au)_i'(t) = \sum_{j=0}^{1} g_{ij}(t) f_j(u_j(0)) + \sum_{j=0}^{1} \int_0^t g_{ij}(t - \tau) f_j'(u_j(\tau))\, u_j'(\tau)\, d\tau.$$

Hence

$$| t^\alpha (Au)_i'(t)| \leqslant 2M_0 t^\alpha \omega(t) + M_1 t^\alpha \int_0^t \omega(t - \tau)\, \tau^{-\alpha}\, d\tau . \| t^\alpha u' \|_{\mathscr{C}}$$

and therefore

$$\| t^\alpha (Au)' \|_{\mathscr{C}} \leqslant 4M_0 h_2(\gamma) + 2M_1 h_3(\gamma) \| t^\alpha u' \|_{\mathscr{C}} .$$

Thus

$$\| Au \|_X \leqslant 4M_0(h_1 + h_2) + 2M_1 h_3 \| u \|_X . \tag{13}$$

Suppose that $R > 0$ and $\| u - \phi \|_X \leqslant R$. Then it follows from (13) that

$$\| Ku - \phi \|_X \leqslant 4M_0(h_1 + h_2) + 2M_1 h_3(\| \phi \|_X + R).$$

Hence, because $h_i(\gamma) \to 0$ and $\| \phi \|_X$ does not increase as $\gamma \to 0+$, there exists $\gamma_0 > 0$ such that $\| Ku - \phi \|_X \leqslant R$ if $\gamma \leqslant \gamma_0$. Thus, if $\gamma \leqslant \gamma_0$, K maps the closed ball

$$\bar{B}_R(\phi) = \{\zeta \in X: \| \zeta - \phi \|_X \leqslant R\}$$

into itself.

We next show that K is a contraction for sufficiently small values of γ. This will be so if A is a contraction for small γ.

Let $u, v \in \bar{B}_R(\phi)$. Then we obtain, using the mean value theorem and the bound for g_{ij}:

$$\begin{aligned} |(Au)_i(t) - (Av)_i(t)| &\leqslant M_1 \sum_{j=0}^{1} \int_0^t \omega(t-\tau) | u_j(\tau) - v_j(\tau) | \, d\tau \\ &\leqslant M_1 h_1(\gamma) \| u - v \|_{\mathscr{C}}, \qquad i = 0, 1, \end{aligned}$$

when $0 < t < \gamma$. Hence

$$\| Au - Av \|_{\mathscr{C}} \leqslant 2M_1 h_1(\gamma) \| u - v \|_{\mathscr{C}}.$$

Similarly, we obtain

$$\begin{aligned} \| t^\alpha (Au)' - t^\alpha (Av)' \|_{\mathscr{C}} \leqslant 2(M_1 h_2 + RM_2 h_3 + \| \phi \|_X M_2 h_3) \| u - v \|_{\mathscr{C}} \\ + 2M_1 h_3 \| t^\alpha u' - t^\alpha v' \|_{\mathscr{C}}. \end{aligned}$$

Therefore

$$\| Au - Av \|_X \leqslant L(\gamma) \| u - v \|_X,$$

where

$$L(\gamma) = \max\{2M_1[h_1(\gamma) + h_2(\gamma)] + 2M_2[R + \| \phi \|_X] h_3(\gamma), 2M_1 h_3(\gamma)\}.$$

Because $h_i(\gamma) \to 0$ and $\| \phi \|_X$ does not increase as $\gamma \to 0+$, $L(\gamma) \to 0$ as $\gamma \to 0+$ and there exists a number $\gamma_1 > 0$ such that $L(\gamma) < 1$ if $\gamma \leqslant \gamma_1$. Thus if $\gamma \leqslant \gamma^* = \min\{\gamma_0, \gamma_1\}$, the operator K is a contraction which maps $\bar{B}_R(\phi)$ into itself.

We now note that the above argument establishes that $\gamma^* < 1$ may be chosen so that, for any ϕ in a bounded set of $X(1)$, K maps $\bar{B}_R(\phi)$ into itself and is a contraction. We also note that if we replace ψ in (9) by $\bar{\psi} = u(\cdot, \tau; \psi)$ for any $\tau \geqslant 0$, then by Lemma 3 and the estimates (11), (12), the corresponding functions $\bar{\phi}$ are bounded in $X(1)$ independently of $\tau \geqslant 0$. Hence the above argument establishes that $u(t) \in \mathscr{C}^1(\tau, \tau + \gamma^*)$ for any $\tau \geqslant 0$, and the desired result follows.

COROLLARY. *Let* $\psi \in C([0, 1])$. *Then* $u_t(\cdot, \cdot; \psi) \in C([0, 1] \times [a, b])$ *for* $0 < a < b < \infty$.

Proof. This is immediate from Lemma 2.

II

Let $u(x, t; \psi)$ be the solution of the problem (4)–(6),

$$\begin{aligned} u_t &= u_{xx}, & (x, t) &\in Q, \\ u_x(i, t) &= (-1)^i f_i(u(i, t)), & i &= 0, 1, \quad t > 0, \\ u(x, 0) &= \psi(x), & 0 &\leqslant x \leqslant 1, \end{aligned}$$

in which the functions f_i satisfy conditions (H1) and (H2), and $\psi \in C([0, 1])$. Define the operators $T(t)$: $C([0, 1]) \to C([0, 1])$ by

$$T(t)\psi = u(\cdot, t; \psi), \qquad t \geqslant 0.$$

It follows from Theorem 1 that $\{T(t)\}$ $t \geqslant 0$ is a semigroup on $C([0, 1])$; that is, (i) $T(0) =$ identity, and (ii) $T(s)\, T(t) = T(s + t)$ for all $s, t \geqslant 0$.

LEMMA 4. *Let $\psi \in C([0, 1])$ and $\tau > 0$. Then the set $\{T(t)\psi: t \geqslant \tau\}$ is precompact in* $C([0, 1])$.

Proof. It was shown in Section I that there is a constant K_0 such that

$$\| T(t)\psi \|_0 \leqslant K_0, \qquad \text{for} \quad t \geqslant 0, \tag{14}$$

where $\| \cdot \|_0$ denotes the supremum norm in $C([0, 1])$. Moreover, by Lemma 3, $u_x(x, t; \psi)$ is uniformly bounded for $x \in [0, 1]$ and $t \geqslant \tau$. Hence the set $\{T(t)\psi: t \geqslant \tau\}$ is bounded and equicontinuous, and thus precompact by Ascoli's theorem.

In the usual way we define the norm of an element $\zeta \in C^1([0, 1])$ by $\| \zeta \|_1 = \| \zeta \|_0 + \| \zeta' \|_0$. We need the following continuity properties of the semigroup $\{T(t)\}$ $t \geqslant 0$.

LEMMA 5. (a) *For $t > 0$, $T(t)$ is a continuous map from $C([0, 1])$ into $C^1([0, 1])$.* (b) *For each $\psi \in C([0, 1])$ the map $T(\cdot)\psi$: $(0, \infty) \to C^1([0, 1])$ is continuous.*

Proof. First note that (b) follows immediately from (a) and the known continuity of $T(\cdot)\psi$: $(0, \infty) \to C([0, 1])$.

To prove (a) let $\psi_n \to \psi$ in $C([0, 1])$. let $t > 0$, and set $u(t) = T(t)\psi$, $u_n(t) = T(t)\psi_n$. Then by Theorem 1,

$$\| u_n(t) - u(t) \|_0 \to 0 \qquad \text{as} \quad n \to \infty$$

and we need to show that

$$\| u_{nx}(\cdot, t) - u_x(\cdot, t) \|_0 \to 0 \qquad \text{as} \quad n \to \infty.$$

It follows from (8) that

$$u_{nx}(x, t) - u_x(x, t) = \int_0^1 G_x(x, \xi, t)\{\psi_n(\xi) - \psi(\xi)\} \, d\xi$$
$$- \sum_{i=0}^{1} \int_0^t G_x(x, i, t - \tau)\{f_i(u_n(i, \tau)) - f_i(u(i, \tau))\} \, d\tau.$$

Let $\tilde{\psi}_n$ and $\tilde{\psi}$ be the periodic extensions of, respectively, ψ_n and ψ into $\mathbb{R}$ such that $\tilde{\psi}_n$ and $\tilde{\psi}$ are symmetric with respect to $x = 0$ and $x = 1$. Then the first term in the expression for $u_{nx} - u_x$ can be written as

$$I_1 = \int_{-\infty}^{\infty} J_x(x - \xi, t)\{\tilde{\psi}_n(\xi) - \tilde{\psi}(\xi)\} \, d\xi.$$

Hence

$$| I_1 | \leqslant \int_{-\infty}^{\infty} | J_x(x - \xi, t)| \, d\xi . \sup_{\mathbb{R}} | \tilde{\psi}_n(\xi) - \tilde{\psi}(\xi)| = (\pi t)^{-1/2} \| \psi_n - \psi \|_0 . \quad (15)$$

We denote the second term in the expression for $u_{nx} - u_x$ by I_2. It follows from the mean value theorem and the estimate for f_i' that for $t \leqslant T$,

$$| I_2 | \leqslant M_1 \sum_{i=0}^{1} \int_0^t | G_x(x, i, t - \tau)| \, d\tau . \max_{\bar{Q}_T} | u_n(x, t) - u(x, t)|$$
$$\leqslant M_1 \sum_{i=0}^{1} \int_0^t | G_x(x, i, t - \tau)| \, d\tau . C_T \| \psi_n - \psi \|_0 . \quad (16)$$

An elementary computation shows that the two integrals in (16) are bounded for $0 \leqslant t \leqslant T$. Hence there exists a constant K such that

$$| I_2 | \leqslant K \| \psi_n - \psi \|_0 , \qquad 0 \leqslant t \leqslant T.$$

This implies, together with the estimate for I_1, that

$$\| u_{nx}(\cdot, t) - u_x(\cdot, t)\|_0 \leqslant \{(\pi t)^{-1/2} + K\} \| \psi_n - \psi_0 \|_0 , \qquad 0 < t \leqslant T,$$

from which the result follows.

Define the function V: $C^1([0, 1]) \to \mathbb{R}$ by

$$V(\zeta) = \tfrac{1}{2} \int_0^1 (\zeta'(x))^2 \, dx + \sum_{i=0}^{1} F_i(\zeta(i)),$$

where

$$F_i(p) = \int_0^p f_i(q) \, dq, \qquad i = 0, 1.$$

It is readily shown that V is continuous.

LEMMA 6. *Let* $\psi \in C([0, 1])$ *and let* $0 < \tau < t < \infty$. *Then*

$$V(T(t)\psi) - V(T(\tau)\psi) = -\int_\tau^t ds \int_0^1 u_t^2(x, s; \psi)\, dx. \tag{17}$$

Proof. Let $0 < \delta < \frac{1}{2}$. Following Chafee [4], define $V_\delta : C^1([0, 1]) \to \mathbb{R}$ by

$$V_\delta(\zeta) = \tfrac{1}{2} \int_\delta^{1-\delta} (\zeta'(x))^2\, dx + F_0(\zeta(0)) + F_1(\zeta(1)).$$

Since $u(\cdot, t; \psi) = T(t)\psi$ satisfies the heat equation, it is smooth in Q. Therefore for $t > 0$,

$$(d/dt)\, V_\delta(T(t)\psi) = u_x u_t \,|_{x=\delta}^{x=1-\delta} - \int_\delta^{1-\delta} u_t^2\, dx + (d/dt)[F_0(u(0, t; \psi)) + F_1(u(1, t; \psi))].$$

Hence if $0 < \tau < t < \infty$,

$$V_\delta(T(t)\psi) - V_\delta(T(\tau)\psi) = \int_\tau^t u_x u_t \,|_{x=\delta}^{x=1-\delta}\, ds - \int_\tau^t ds \int_\delta^{1-\delta} u_t^2\, dx + [F_0(u(0, s; \psi)) + F_1(u(1, s; \psi))]_{s=\tau}^{s=t}.$$

Let $\delta \to 0+$. Then by Lemma 2 and the dominated convergence theorem we obtain (17).

For $\psi \in C([0, 1])$ define the ω-limit set $\omega(\psi)$ by $\omega(\psi) = \{\chi \in C^1([0, 1])$: there exists $\{t_n\}$, $t_n \to \infty$ as $n \to \infty$, with $T(t_n)\psi \to \chi$ in $C^1([0, 1])\}$. We can now prove our main result:

THEOREM 3. *Let* $\psi \in C([0, 1])$. *Then, as* $t \to \infty$, $T(t)\psi \to v$ *in* $C^1([0, 1)]$, *where* v *is an equilibrium solution.*

Proof. Without loss of generality we may assume that $\psi \in C^1([0, 1])$. By Lemma 5 $\{T(t)\}t \geq 0$ defines by restriction a semigroup of continuous operators on $C^1([0, 1])$ such that for every $\psi \in C^1([0, 1])$ the map $T(.)\psi$: $(0, \infty) \to C^1([0, 1])$ is continuous.

Since $u(x, t; \psi)$ is bounded, $V(T(t)\psi)$ is bounded below for $t \geq 0$. Also, by Lemma 6, $V(T(t)\psi)$ is nonincreasing for $t > 0$. Bearing in mind Lemma 4 it follows from [9] that $\omega(\psi)$ is nonempty, positively invariant (i.e., $T(t)\, \omega(\psi) \subseteq \omega(\psi)$ for $t \geq 0$) and connected. Furthermore, as $t \to \infty$, $d(T(t)\psi, M) \to 0$, where d denotes distance in $C^1([0, 1])$ and where M is the largest positively invariant set contained in $\{\chi \in C^1([0, 1])$: $V(\chi) = \inf_{t>0} V(T(t)\psi)\}$.

By Lemma 6, M contains only equilibrium solutions. Since these solutions are by hypothesis (H2) isolated, and since $\omega(\psi)$ is connected, it follows that $\omega(\psi) = \{v\}$ for some equilibrium solution v, and that $T(t)\psi \to v$ in $C^1([0, 1])$ as $t \to \infty$.

Remark. In [9] it was assumed (partly so as to obtain stronger conclusions than we require) that the map $(t, \psi) \to T(t)\psi$ is jointly continuous on $(0, \infty) \times C^1([0, 1])$, whereas we have established only separate continuity with respect to t and ψ. This apparent restriction was removed by Dafermos [7]; Chernoff and Marsden [6] have shown, however, that for a semigroup defined on a metric space joint continuity is implied by separate continuity. For our problem joint continuity is easy to prove directly.

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